Parallelizing The Matrix Multiplication

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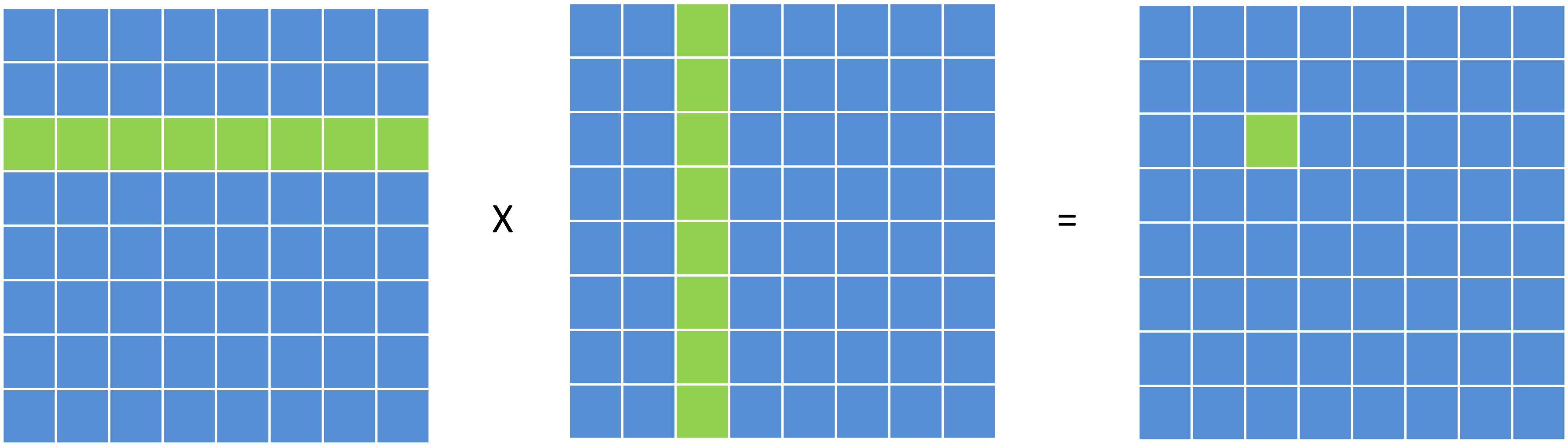
Serial version



$$A_{md} \quad \text{x} \quad B_{dn} \quad = \quad C_{mn}$$

$$c_{i,j} = \sum_{k=1}^{d} a_{i,k} \cdot b_{k,j}$$

//Read and validate command line arguments
Read M,N,D from the command line

//Initialize the arrays
For all elements of A and B
    initial value = function (i , j)

// Matrix multiplication
For each C (i,j)
// Take the inner product of row i of A and column j of B
    For each A (i,k) and B(k,j)
        C (i,j) = C (i,j) + A (i,k) * B(k,j)

$$a_{i,j} = i + j$$
$$b_{i,j} = i * j$$

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Anything else?

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How do we know the result is correct?

Result validation is very important!

$$c_{i,j} = \sum_{k=1}^{N} a_{i,k} \cdot b_{k,j}$$

$$= \sum_{k=1}^{N} (i+k) \cdot j \cdot k$$

$$= \sum_{k=1}^{N} (ijk + jk^2)$$

$$= ij \sum_{k=1}^{N} k + j \sum_{k=1}^{N} k^2$$

$$\boxed{= ij \cdot \frac{N(N+1)}{2} + j \cdot \frac{N(N+1)(2N+1)}{6}}$$

Assuming A, B and C are all N x N square matrices

We can use this formula to validate the result from the program

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//Validate the result
Print out an element of C and compare it to the value given by the formula

$$a_{i,j} = i + j$$
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Serial Program Details

- A, B and C are all N x N square matrices
- Take input arguments
  - N: dimension of the matrices
  - i_{peek}, j_{peek}: indices of the element used for result validation
- Values of A and B:

$$a_{i,j} = i + j$$

$$b_{i,j} = i * j$$

- Output
  - How much time it takes (performance measurement)
  - The value of C(i_{peek}, j_{peek}) and the value given by the formula

Parallel version

Writing a parallel program step by step

- Step 1. Start from serial programs as a baseline
  – Something to check correctness and efficiency against
- Step 2. Analyze and profile the serial program
  – Identify the "hotspot"
  – Identify the parts that can be parallelized
- Step 3. Parallelize code incrementally
- Step 4. Check correctness of the parallel code
- Step 5. Iterate step 3 and 4

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Which parts can and should be parallelized?

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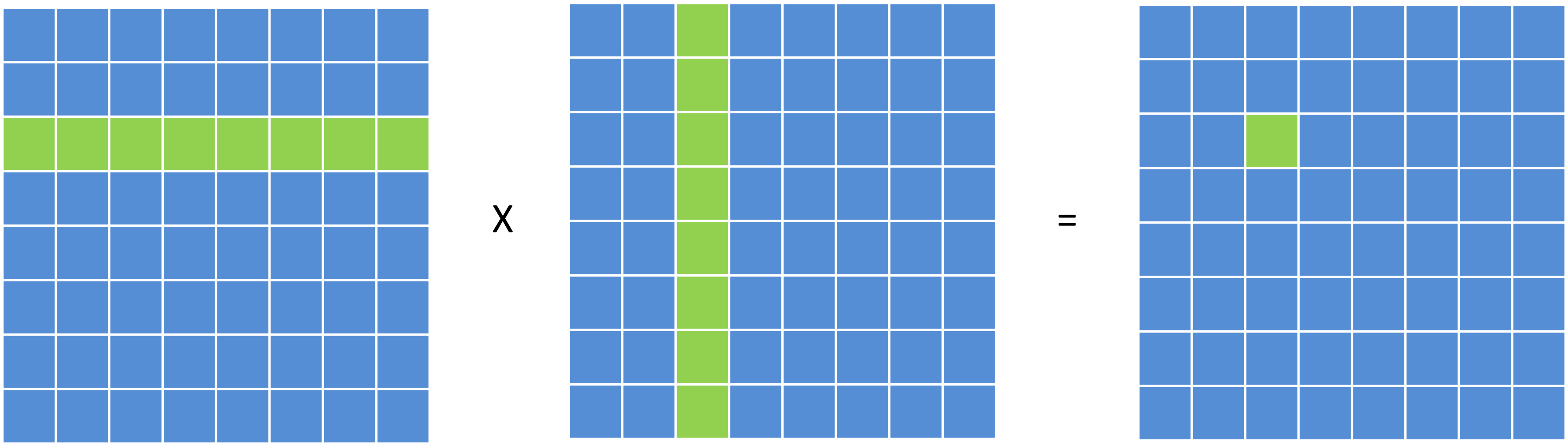

A_{nn} x B_{nn} = C_{nn}

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} \cdot b_{k,j}$$

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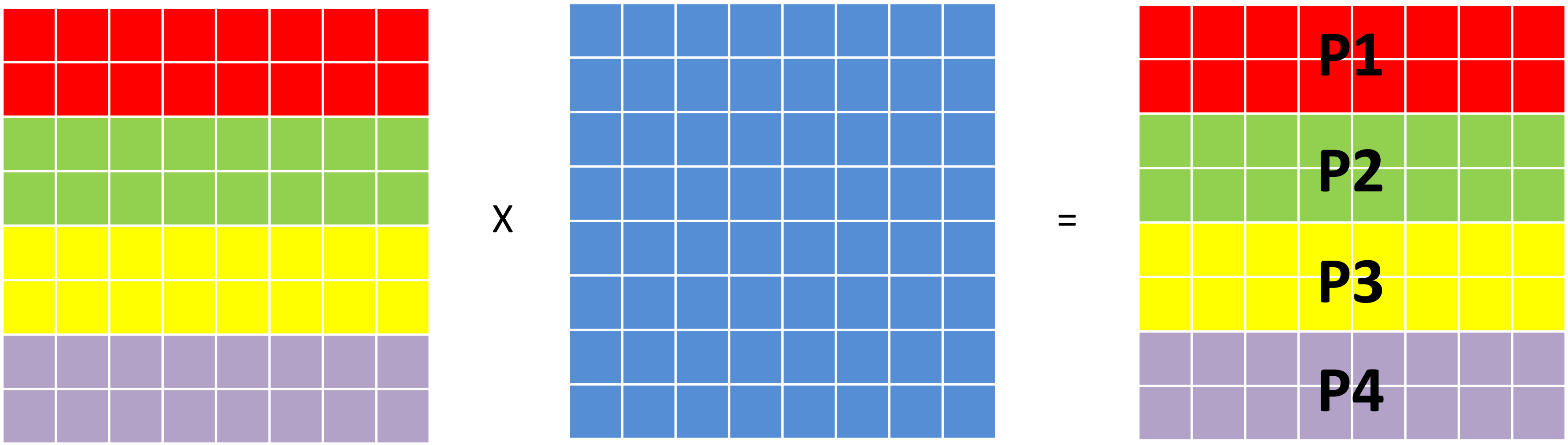
Some Considerations

- Decomposition
  - 1-D or 2-D
- Data distribution
  - Each process owns the entire matrices, but only performs calculation on a part of them, or
  - Each process only owns the sub-matrices that it is going to process



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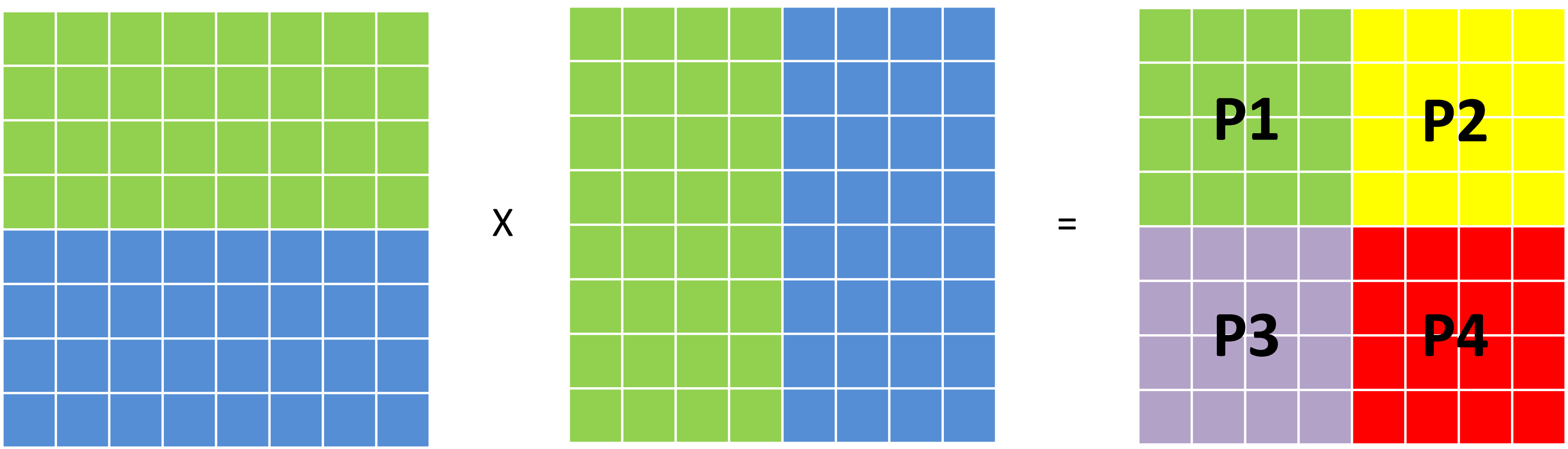

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$$c_{i,j} = \sum_{k=1}^{N} a_{i,k} \cdot b_{k,j}$$

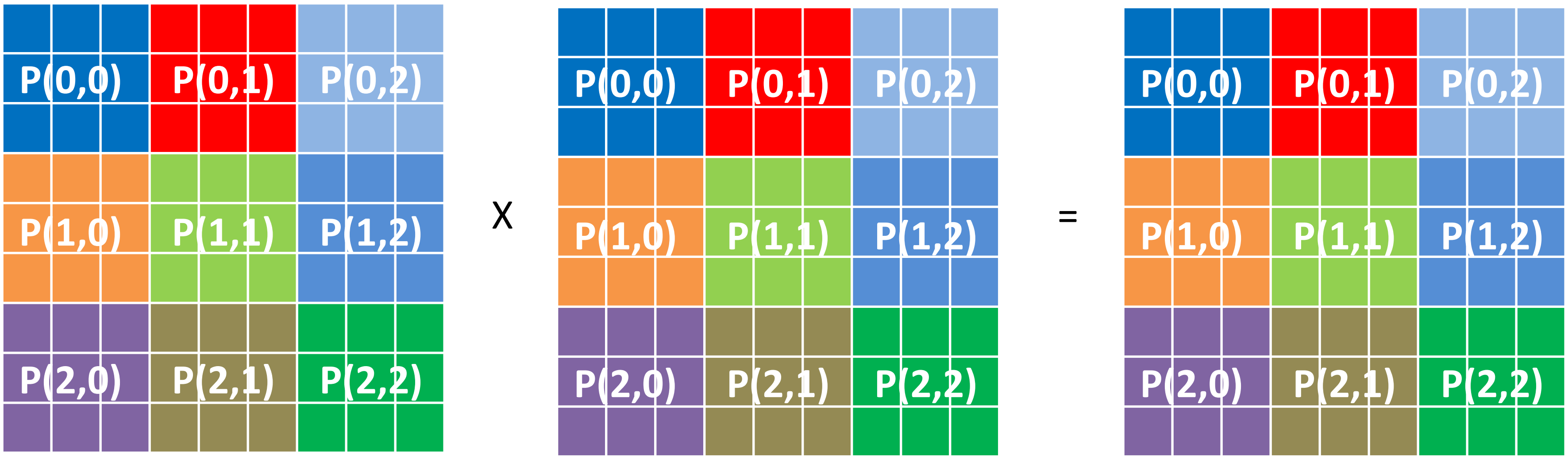
Optimization Conderations

- Avoid data movement as much as possible (i.e. increase the amount of computation done relative to the amount of data moved)
  - True for both serial and parallel programs
  - For parallel programs, data movement means data communication between host and device (GPU) or between different nodes (distributed memory systems)
- Reduce the memory footprint
- When developing a parallel program, it’s important to know what to expect in terms of speedup and scaling

Cannon's Algorithm

Cannon's Algorithm

- Assume
  - the number of processes *p* is a perfect square
  - the matrices are N x N square
- Arrange the processes into a 2-D $\sqrt{p}$ x $\sqrt{p}$ process grid
  - For each matrix, each process is assigned with a block of $N/\sqrt{p}$ x $N/\sqrt{p}$



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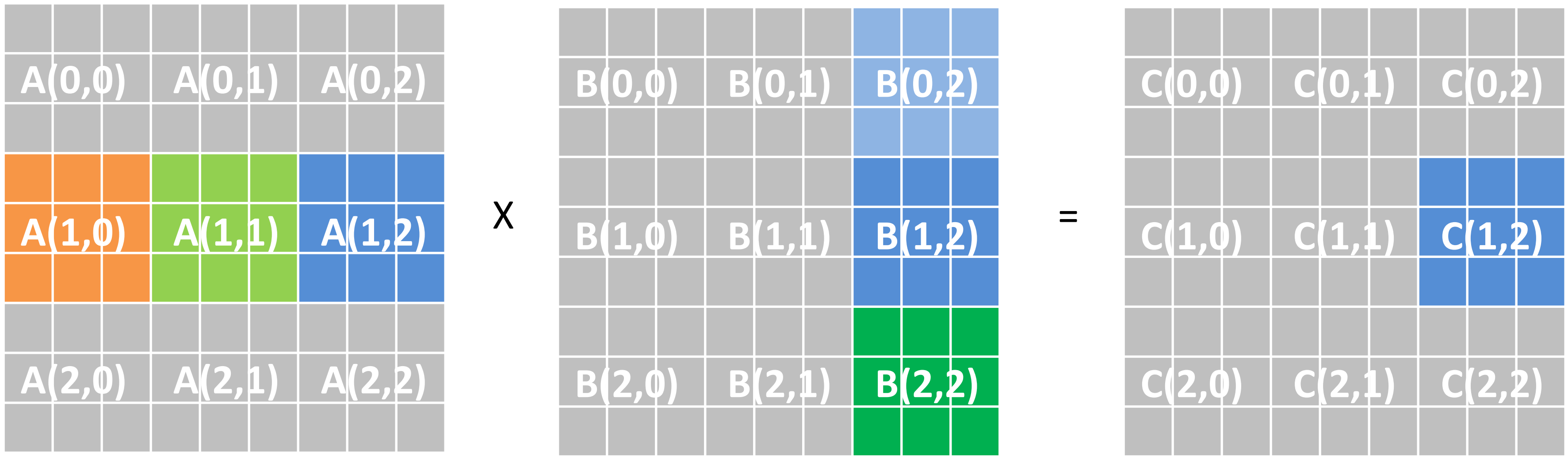
Working by elements



A_{nn} x B_{nn} = C_{nn}

C(1,2) = A(1,0) X B (0,2) + A(1,1) X B(1,2) + A(1,2) X B(2,2)

Working by blocks

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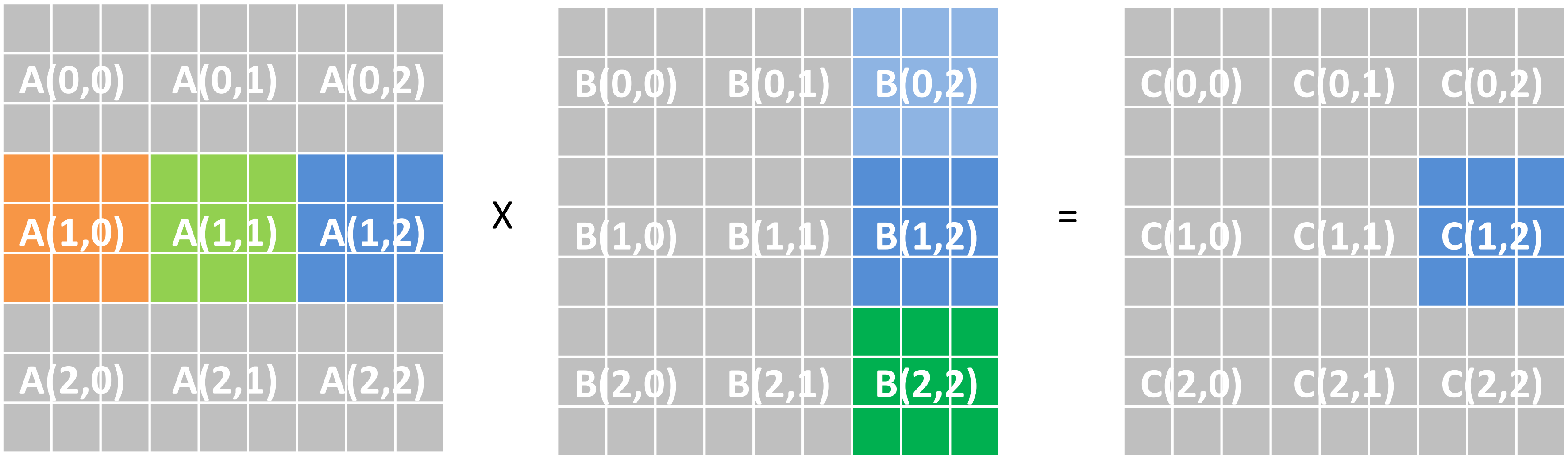

$$A_{nn} \quad x \quad B_{nn} \quad = \quad C_{nn}$$

$$C(1,2) = \underline{A(1,0) \text{ X } B(0,2)} + \underline{A(1,1) \text{ X } B(1,2)} + \underline{A(1,2) \text{ X } B(2,2)}$$

Three phases: phase1 phase2 phase3

The multiplication is completed in $\sqrt{p}$ phases

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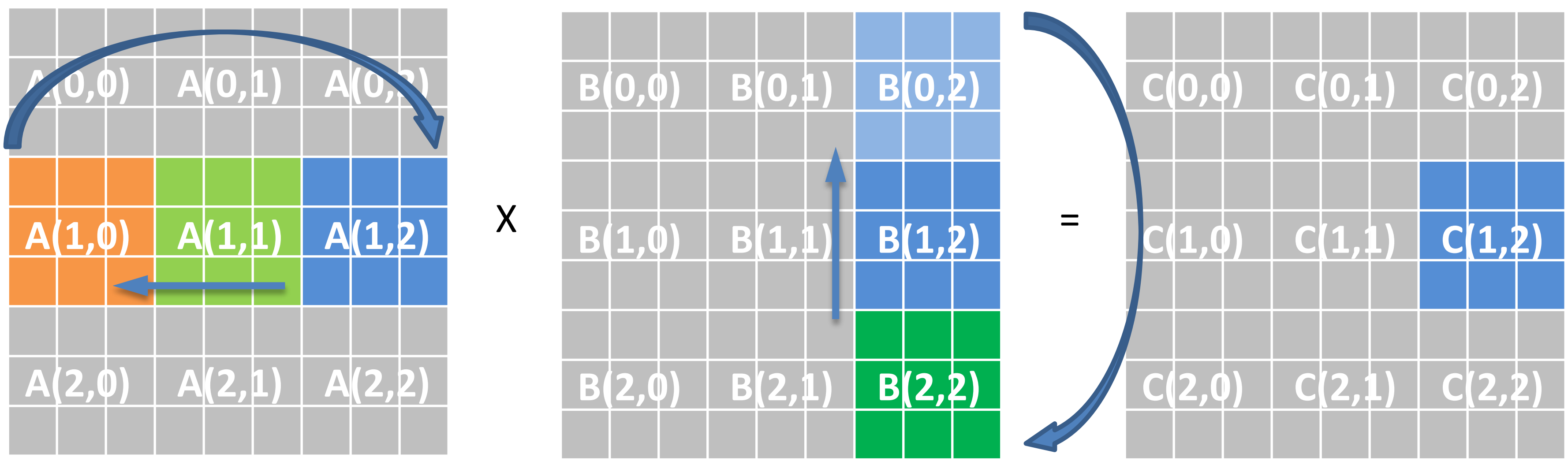
Cannon's Algorithm

- Two stages
  - Skew the matrices so everything aligns properly
    - Shift row i of A by i columns to the left
    - Shift column j of B by j rows to the up
  - Shift and multiply
    - Each process calculate the local product and add into the accumulated sum
    - Shift A by 1 column to the left
    - Shift B by 1 row to the up
    - Repeat $\sqrt{\boldsymbol{p}}$ times
- All shifts wrap around (circular)



A_{nn} x B_{nn} = C_{nn}

Process P(1,2) owns	After initialization:	A(1,2), B(1,2), C(1,2)
	After skewing:	A(1,0), B(0,2), C(1,2)
	Shifting once:	A(1,1), B(1,2), C(1,2)
	Shifting twice:	A(1,2), B(2,2), C(1,2)

Cannon’s Algorithm – Pseudo Code

```
For i = 0 to sqrt(p) – 1
    Shift A(i,:) to the left by i
For j = 0 to sqrt(p) – 1
    Shift B(:,j) to the up by j
For k = 0 to sqrt(p) – 1
    For i = 0 to sqrt(p) – 1 and j = 0 to sqrt(p) – 1
        C(i,j) += A(i,j) X B(i,j)
        Shift A(i,:) to the left by 1
        Shift B(:,j) to the up by 1
```