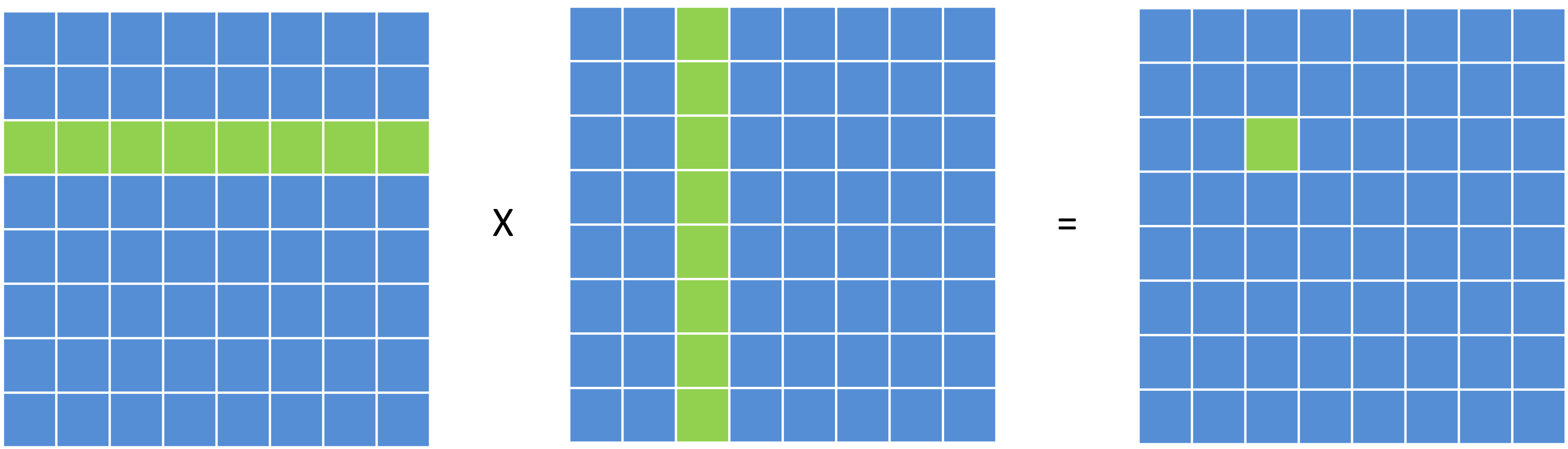
Parallelizing The Matrix Multiplication

CENTER FOR COMPUTATION & TECHNOLOGY

Serial version



A_{md} x B_{dn} = C_{mn}

$$c_{i,j} = \sum_{k=1}^{d} a_{i,k} \cdot b_{k,j}$$

//Read and validate command line arguments
Read M,N,D from the command line

//Initialize the arrays
For all elements of A and B
  initial value = function (i , j)

// Matrix multiplication
For each C (i,j)
// Take the inner product of row i of A and column j of B
  For each A (i,k) and B(k,j)
    C (i,j) = C (i,j) + A (i,k) * B(k,j)

$$a_{i,j} = i + j$$
$$b_{i,j} = i * j$$

```
//Read and validate command line arguments
Read M,N,D from the command line

//Initialize the arrays
For all elements of A and B
   initial value = function ( i , j )

// Matrix multiplication
For each C (i,j)
// Take the inner product of row i of A and column j of B
   For each A (i,k) and B(k,j)
      C (i,j) = C (i,j) + A (i,k) * B(k,j)
```

$$a_{i,j} = i + j$$
$$b_{i,j} = i * j$$

Anything else?

```
//Read and validate command line arguments
Read M,N,D from the command line

//Initialize the arrays
For all elements of A and B
   initial value = function ( i , j )

// Matrix multiplication
For each C (i,j)
// Take the inner product of row i of A and column j of B
   For each A (i,k) and B(k,j)
      C (i,j) = C (i,j) + A (i,k) * B(k,j)
```

$$a_{i,j} = i + j$$
$$b_{i,j} = i * j$$

How do we know the result is correct?

Result validation is very important!

$$c_{i,j} = \sum_{k=1}^{N} a_{i,k} \cdot b_{k,j}$$

$$= \sum_{k=1}^{N} (i + k) \cdot j \cdot k$$

$$= \sum_{k=1}^{N} (ijk + jk^2)$$

$$= ij \sum_{k=1}^{N} k + j \sum_{k=1}^{N} k^2$$

$$\boxed{= ij \cdot \frac{N(N+1)}{2} + j \cdot \frac{N(N+1)(2N+1)}{6}}$$

Assuming A, B and C are all N x N square matrices

We can use this formula to validate the result from the program

//Read and validate command line arguments
Read M,N,D from the command line

//Initialize the arrays
For all elements of A and B
initial value = function (i , j)

$$a_{i,j} = i + j$$
$$b_{i,j} = i * j$$

// Matrix multiplication
For each C (i,j)
// Take the inner product of row i of A and column j of B
For each A (i,k) and B(k,j)
C (i,j) = C (i,j) + A (i,k) * B(k,j)

//Validate the result
Print out an element of C and compare it to the value given by the formula

CENTER FOR COMPUTATION & TECHNOLOGY

Serial Program Details

- A, B and C are all N x N square matrices
- Take input arguments
  - N: dimension of the matrices
  - i_{peek}, j_{peek}: indices of the element used for result validation
- Values of A and B:

$$a_{i,j} = i + j$$

$$b_{i,j} = i * j$$

- Output
  - How much time it takes (performance measurement)
  - The value of C(i_{peek}, j_{peek}) and the value given by the formula

CENTER FOR COMPUTATION & TECHNOLOGY

Parallel version

Writing a parallel program step by step

- Step 1. Start from serial programs as a baseline
  - Something to check correctness and efficiency against
- Step 2. Analyze and profile the serial program
  - Identify the “hotspot”
  - Identify the parts that can be parallelized
- Step 3. Parallelize code incrementally
- Step 4. Check correctness of the parallel code
- Step 5. Iterate step 3 and 4

//Read and validate command line arguments
Read M,N,D from the command line

//Initialize the arrays
For all elements of A and B
    initial value = function (i , j)

// Matrix multiplication
For each C (i,j)
// Take the inner product of row i of A and column j of B
    For each A (i,k) and B(k,j)
        C (i,j) = C (i,j) + A (i,k) * B(k,j)

//Validate the result
Print out an element of C and compare it to the value given by the formula

Which parts can and should be parallelized?

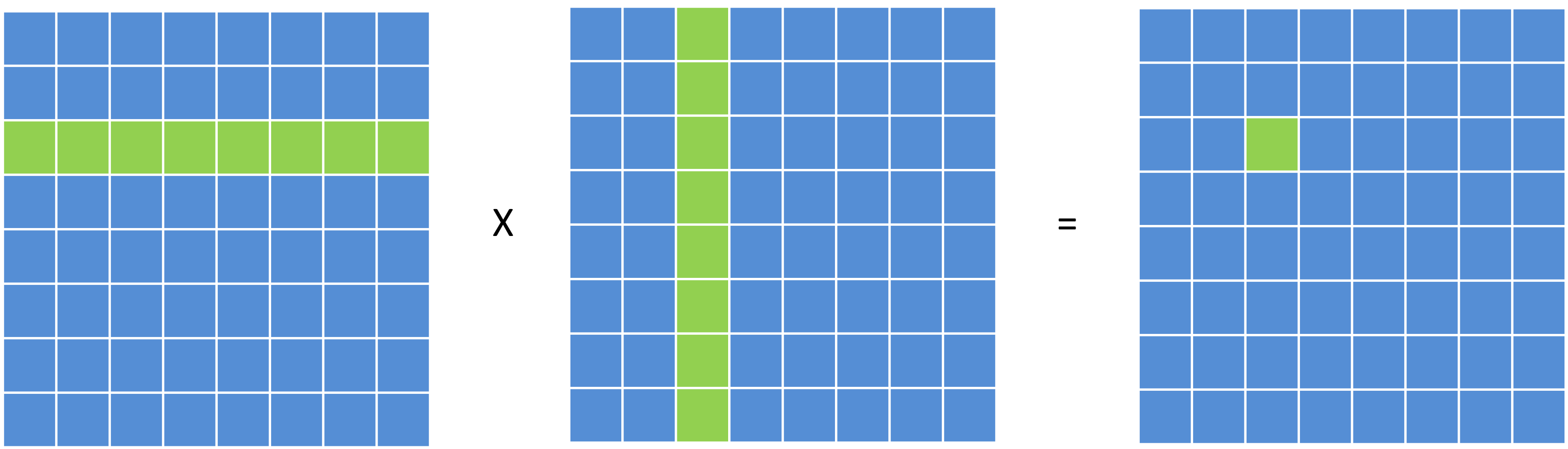
```
//Read and validate command line arguments
Read M,N,D from the command line

//Initialize the arrays
For all elements of A and B
   initial value = function ( i , j )

// Matrix multiplication
For each C (i,j)
// Take the inner product of row i of A and column j of B
   For each A (i,k) and B(k,j)
      C (i,j) = C (i,j) + A (i,k) * B(k,j)

//Validate the result
Print out an element of C and compare it to the value
given by the formula
```

Which parts can and should be parallelized?

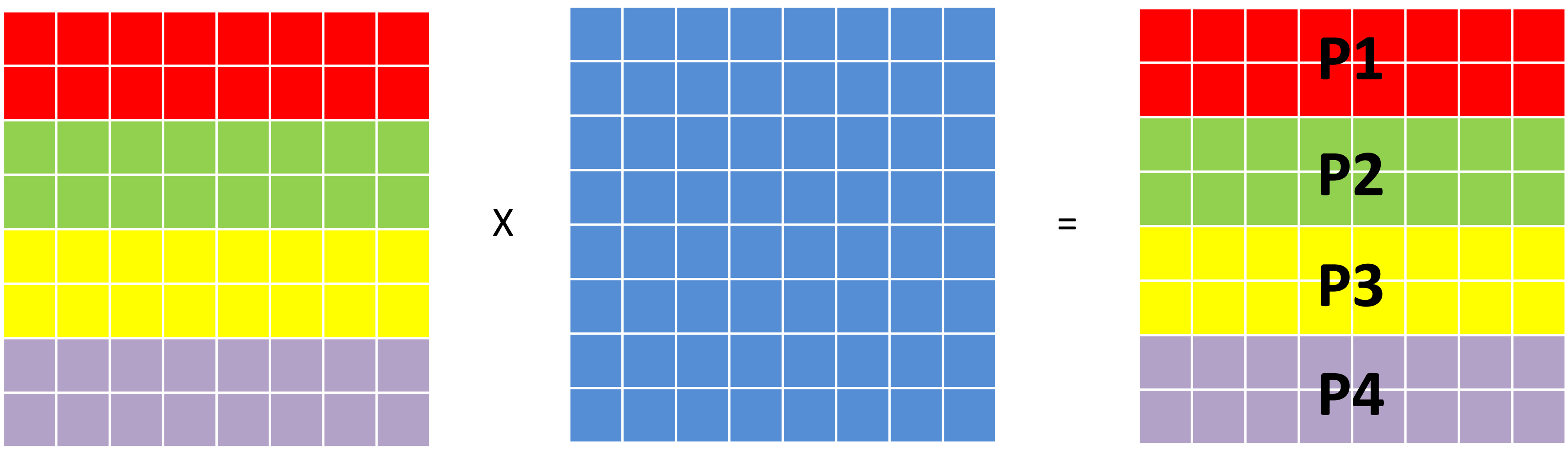


A_{nn} x B_{nn} = C_{nn}

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} \cdot b_{k,j}$$

Some Considerations

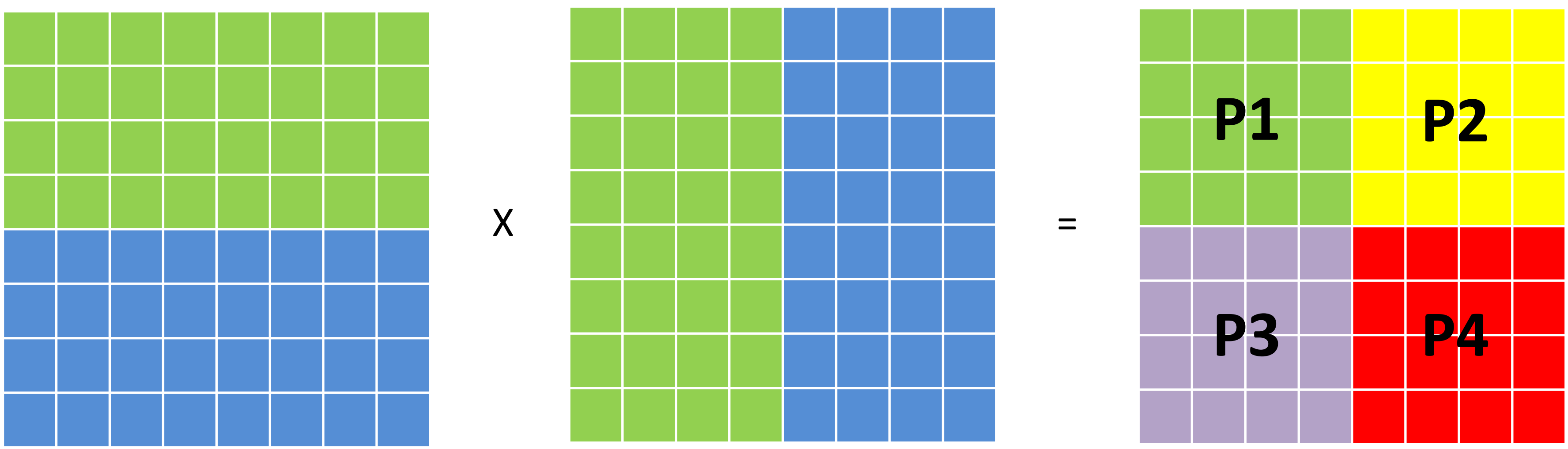
- Decomposition
  - 1-D or 2-D
- Data distribution
  - Each process owns the entire matrices, but only performs calculation on a part of them, or
  - Each process only owns the sub-matrices that it is going to process



A_{nn} x B_{nn} = C_{nn}

$$c_{i,j} = \sum_{k=1}^{N} a_{i,k} \cdot b_{k,j}$$

CENTER FOR COMPUTATION & TECHNOLOGY



A_{nn} x B_{nn} = C_{nn}

$$c_{i,j} = \sum_{k=1}^{N} a_{i,k} \cdot b_{k,j}$$

CENTER FOR COMPUTATION & TECHNOLOGY

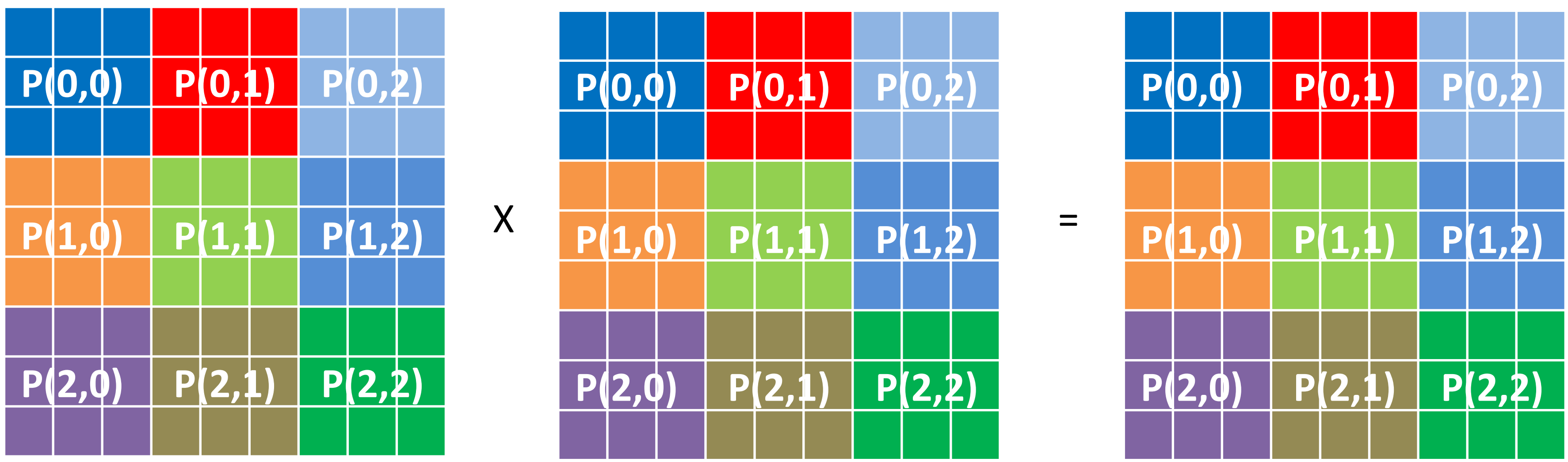
Optimization Conderations

- Avoid data movement as much as possible (i.e. increase the amount of computation done relative to the amount of data moved)
  - True for both serial and parallel programs
  - For parallel programs, data movement means data communication between host and device (GPU) or between different nodes (distributed memory systems)
- Reduce the memory footprint
- When developing a parallel program, it's important to know what to expect in terms of speedup and scaling

Cannon's Algorithm

Cannon's Algorithm

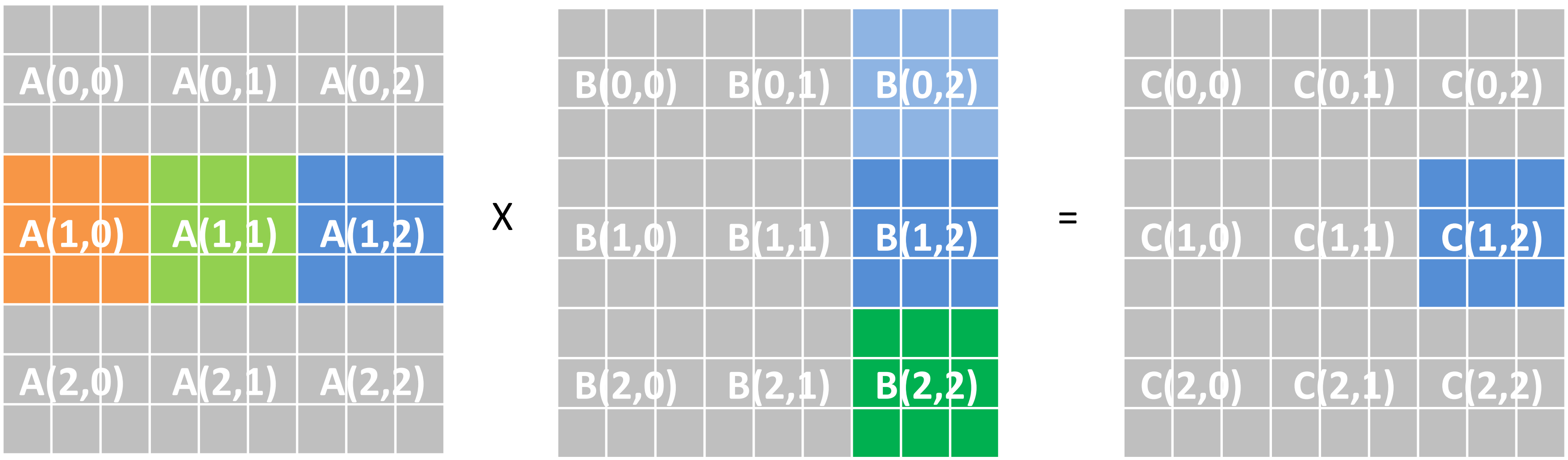
- Assume
  - the number of processes *p* is a perfect square
  - the matrices are N x N square
- Arrange the processes into a 2-D $\sqrt{p}$ x $\sqrt{p}$ process grid
  - For each matrix, each process is assigned with a block of $N/\sqrt{p}$ x $N/\sqrt{p}$



A_{nn} x B_{nn} = C_{nn}

$$c_{i,j} = \sum_{k=1}^{N} a_{i,k} \cdot b_{k,j}$$

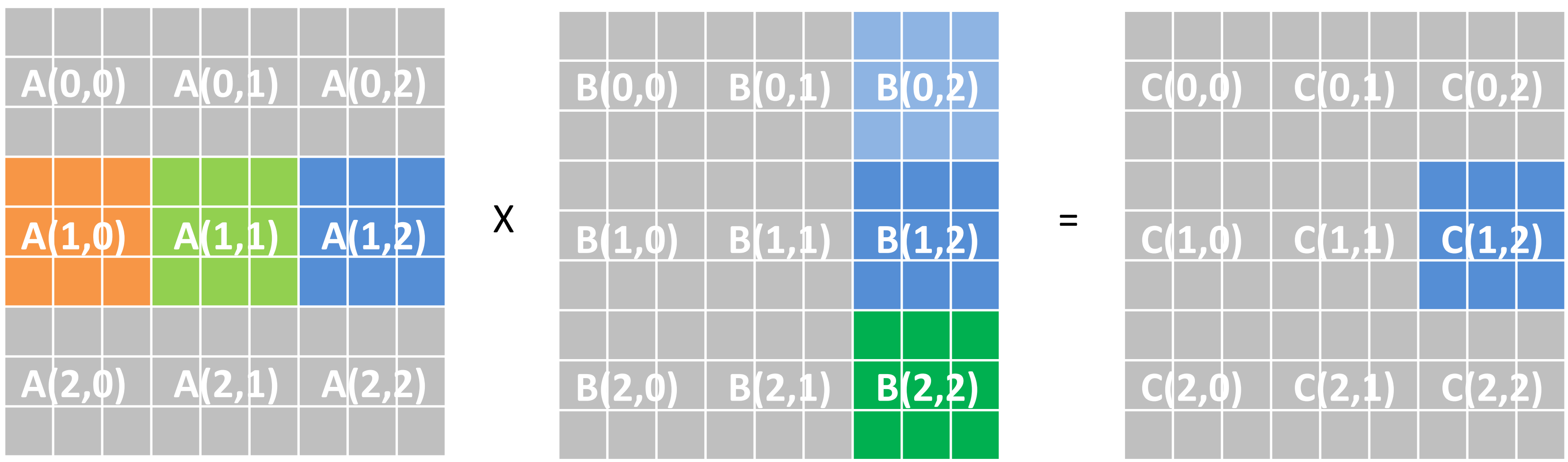
Working by elements



A_{nn} x B_{nn} = C_{nn}

C(1,2) = A(1,0) X B (0,2) + A(1,1) X B(1,2) + A(1,2) X B(2,2)

Working by blocks



A_{nn} x B_{nn} = C_{nn}

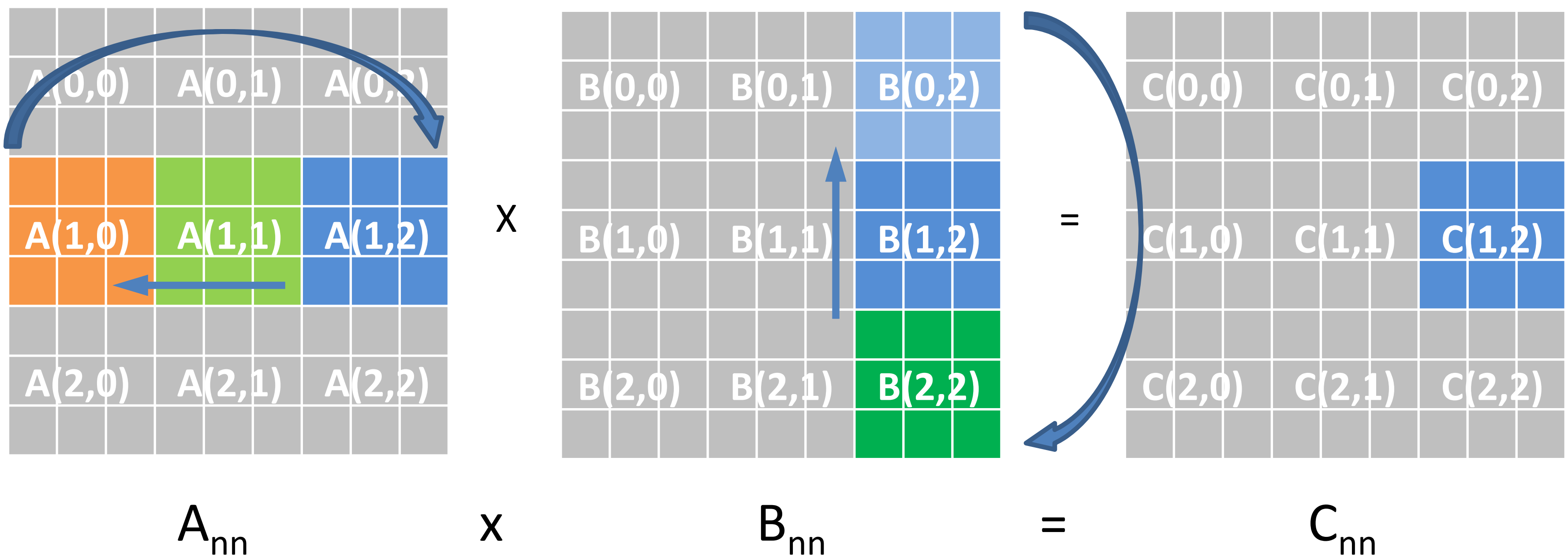
C(1,2) = $\underbrace{\text{A(1,0) X B (0,2)}}_{\text{phase1}}$ + $\underbrace{\text{A(1,1) X B(1,2)}}_{\text{phase2}}$ + $\underbrace{\text{A(1,2) X B(2,2)}}_{\text{phase3}}$

Three phases: phase1 phase2 phase3

The multiplication is completed in $\sqrt{p}$ phases

Cannon's Algorithm

- Two stages
  - Skew the matrices so everything aligns properly
    - Shift row i of A by i columns to the left
    - Shift column j of B by j rows to the up
  - Shift and multiply
    - Each process calculate the local product and add into the accumulated sum
    - Shift A by 1 column to the left
    - Shift B by 1 row to the up
    - Repeat $\sqrt{p}$ times
- All shifts wrap around (circular)



Process P(1,2) owns		
	After initialization:	A(1,2), B(1,2), C(1,2)
	After skewing:	A(1,0), B(0,2), C(1,2)
	Shifting once:	A(1,1), B(1,2), C(1,2)
	Shifting twice:	A(1,2), B(2,2), C(1,2)

CENTER FOR COMPUTATION & TECHNOLOGY

Cannon’s Algorithm – Pseudo Code

```
For i = 0 to sqrt(p) – 1
   Shift A(i,:) to the left by i
For j = 0 to sqrt(p) – 1
   Shift B(:,j) to the up by j
For k = 0 to sqrt(p) – 1
   For i = 0 to sqrt(p) – 1 and j = 0 to sqrt(p) – 1
      C(i,j) += A(i,j) X B(i,j)
      Shift A(i,:) to the left by 1
      Shift B(:,j) to the up by 1
```